

On the existence of compacta of minimal capacity in the theory of rational approximation of multi-valued analytic functions

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Аннотация

For an interval $E = [a, b]$ on the real line, let μ be either the equilibrium measure, or the normalized Lebesgue measure of E , and let V^μ denote the associated logarithmic potential. In the present paper, we construct a function f which is analytic on E and possesses four branch points of second order outside of E such that the family of the admissible compacta of f has no minimizing elements with regard to the extremal theoretic-potential problem, in the external field equals $V^{-\mu}$.

Bibliography: 35 items.

rational approximants, Padé approximation, orthogonal polynomials, distribution of poles, convergence in capacity

Dedicated to the memory of Andrei Aleksandrovich Gonchar and Herbert Stahl

1 Notations

Throughout the paper, we use the following notations.

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$M(K)$ – the space of all positive unit Borel measures μ with supports $S(\mu) = \text{supp } \mu \subset K$, where K is a compact set, $K \subset \overline{\mathbb{C}}$.

δ_z – the Dirac measure at a point z .

For a finite set $E = \{e_1, \dots, e_n\}$ of points (counted with their multiplicities) on $\overline{\mathbb{C}}$, we introduce the measure $\delta_E := \sum_{k=1}^n \delta_{e_k}$.

For a polynomial Q of degree n , we denote $\delta_Q := \delta_{\{z_1, \dots, z_n\}}$, where $\{z_1, \dots, z_n\}$ are the zeros of Q .

We define a spherically normalized potential \mathcal{V}^μ of the measure μ by

$$\mathcal{V}^\mu(z) = \int_{|t| \leq 1} \log \frac{1}{|z - t|} d\mu(t) + \int_{|t| > 1} \log \frac{1}{|1 - z/t|} d\mu(t). \quad (1)$$

$\text{cap}_\psi K$ denotes the capacity of the compactum K in the presence of the harmonic external field ψ (the so-called ψ -weighted capacity). It is known [28] that $\text{cap}_\psi K$ coincides with the transfinite diameter $d_\psi K$ of K in the field ψ , that is

$$d_\psi K := \lim_{n \rightarrow \infty} \left(\max_{z_1, \dots, z_n \in K} \prod_{1 \leq q < r \leq n} |z_q - z_r| e^{-(\psi(z_q) + \psi(z_r))} \right)^{\frac{2}{(n-1)n}}. \quad (2)$$

Let $\psi = \mathcal{V}^{-\mu}$, $S(\mu) \cap K = \emptyset$. For the sake of simplicity we write $\text{cap}_\mu K := \text{cap}_{\mathcal{V}^{-\mu}} K$.

Consider a sequence of compacta $\{K_n\}$, $n = 1, 2, \dots$. It is known [26] that if each K_n is a union of a finite number of continua (the number does not depend on the index n), such that $\lim_{n \rightarrow \infty} K_n = K$ (in the Hausdorff metric, see (41)–(42) below) and if $S(\mu) \cap K = \emptyset$, then

$$\lim_{n \rightarrow \infty} \text{cap}_\mu K_n = \text{cap}_\mu K. \quad (3)$$

$\text{cap } K$ denotes the standard capacity of the compactum $K \subset \mathbb{C}$ in the absence of an external field, i.e. when $\psi \equiv 0$. Since $\mathcal{V}^{\delta_\infty}(z) = 0$ for all $z \in \mathbb{C}$, then $\text{cap } K = \text{cap}_{\delta_\infty} K$.

Given an open set Ω , we say that a sequence of functions $\{R_n\}_{n=1}^\infty$ converges in capacity to the function f on compact subsets of Ω , if for every $K \subset \Omega$ and each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \text{cap}\{z \in K : |(R_n - f)(z)| > \varepsilon\} = 0; \quad (4)$$

we use the notation

$$R_n \xrightarrow{\text{cap}} f, \quad z \in \Omega, \quad n \rightarrow \infty.$$

The notation $\mu_n \xrightarrow{*} \mu$ is used for weak convergence of a sequence of measures μ_n to the measure μ as $n \rightarrow \infty$. Recall that

$$\mu_n \xrightarrow{*} \mu \quad \text{as } n \rightarrow \infty \iff \lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu, \quad (5)$$

for each function f continuous on $\overline{\mathbb{C}}$. Also recall that from each sequence of unit measures one can extract a convergent subsequence.

We denote by $g_K(z, \zeta)$ Green's function of the complement $\overline{\mathbb{C}} \setminus K$ of the compactum K , with a pole at $\zeta \in \overline{\mathbb{C}} \setminus K$. (We let $g_K(z, \zeta) = 0$, if the points z and ζ are contained in two separate components of $\overline{\mathbb{C}} \setminus K$.)

$G_K^\mu(z) = \int g_K(z, t) d\mu(t)$ is Green's potential of the measure μ whose support $S(\mu)$ does not intersect the compactum K .

Now let $S(\mu) \cap K = \emptyset$. Let us remind the notion of the balayage $\tilde{\mu}_K$ of a measure μ onto the compactum K . By definition, the balayage is the unique measure in the space $M(K)$ for which the equality

$$G_K^\mu(z) = \mathcal{V}^{\mu - \tilde{\mu}_K}(z) + w_K^\mu, \quad z \in \overline{\mathbb{C}}, \quad (6)$$

holds quasi everywhere, i.e. except for a set of zero inner capacity; w_K^μ is a constant. In the particular case when $\mu = \delta_\infty$, from equality (6) we obtain that

$$g_K(z, \infty) = \mathcal{V}^{-\lambda_K}(z) + w_K, \quad z \in \overline{\mathbb{C}}, \quad (7)$$

where λ_K is the equilibrium measure of the compact set K (the balayage of the measure $\mu = \delta_\infty$ onto K) and w_K is the Robin constant for K .

We denote \mathcal{F} the class of compacta F of the form $F = \bigcup_{s=1}^p \overline{\gamma}_s$, where each γ_s is an open analytic curve such that the set $\overline{\mathbb{C}} \setminus \overline{\gamma}_s$ is connected and $\gamma_s \cap \overline{\gamma}_t = \emptyset$ for every $s, t = 1, \dots, p$, $s \neq t$. If $F = \bigcup_{s=1}^p \overline{\gamma}_s \in \mathcal{F}$, then we use the notation F_0 for the set $\bigcup_{s=1}^p \gamma_s$.

Assume that $S(\mu) \cap F = \emptyset$. We say that the compactum F is *symmetric in the external field $\mathcal{V}^{-\mu}$* , if $F \in \mathcal{F}$ and the equality

$$\frac{\partial G_F^\mu(z)}{\partial n_+} = \frac{\partial G_F^\mu(z)}{\partial n_-}, \quad z \in F_0, \quad (8)$$

holds, where the normal vectors n_+ and n_- are taken in opposite directions on F . In other words, K is said to be an *S-compactum* in this field.

For the purpose of the present paper, there is no need to provide the general concept of symmetry of a compactum located in an arbitrary harmonic external field ψ (cf. [10]).

\mathcal{E}_m stands for the class of compacta $E \subset \overline{\mathbb{C}}$ of the form $E = \bigsqcup_{j=1}^m E_j$, where E_1, \dots, E_m are pairwise disjoint continua in $\overline{\mathbb{C}}$ (some of the latter may consists of a single point).

$\mathcal{A}(E)$ represents the class of functions f , defined on $E = \bigsqcup_{j=1}^m E_j$ such that each restriction $f_j = f|_{E_j}$, $j = 1, \dots, m$, is holomorphic on E_j and admits an analytic continuation along every path in $\overline{\mathbb{C}}$, not passing through a finite point set A_{f_j} , where A_{f_j} contains at least one branch point of f_j .

Set $A_f = \bigcup_{j=1}^m A_{f_j}$.

Let $E = \bigsqcup_{j=1}^m E_j \in \mathcal{E}_m$ and $f \in \mathcal{A}(E)$. We denote by $\mathfrak{K}_{E,f}$ the family of compacta $K \subset \overline{\mathbb{C}}$ such that $K \cap E = \emptyset$, and each function $f_j = f|_{E_j}$ is holomorphic (i.e., analytic and single-valued) in that connected component D_j of $\overline{\mathbb{C}} \setminus K$, which contains E_j ; $f_j = f_k$ if $D_j = D_k$. We shall refer to these compacta as admissible compacta for the function $f \in \mathcal{A}(E)$.

2 Statement of the problem and discussion

Let $E \in \mathcal{E}_m$, $f \in \mathcal{A}(E)$, and $\mu \in M(E)$. Recently, special attention has been paid to the existence problem of an admissible compactum F minimizing the weighted capacity; i.e., whether there exists $F \in \mathfrak{K}_{E,f}$ such that

$$\text{cap}_\mu F = \inf_{K \in \mathfrak{K}_{E,f}} \text{cap}_\mu K. \quad (9)$$

If the problem is solvable, then we say that the class of compacta $\mathfrak{K}_{E,f}$ contains a minimizing element with respect to the measure μ (or, equivalently, to the field $\mathcal{V}^{-\mu}$). Otherwise, there is no minimizing element with respect to μ . If such an element exists, for brevity let us call it a ‘ μ -minimizing element’. In the particular case, when

$$E = \{\infty\}, \quad \mu = \delta_\infty, \quad f(z) = \sqrt[q]{(z - a_1) \cdots (z - a_q)} - z, \quad f \in H(\{\infty\}), \quad (10)$$

the above mentioned problem coincides with the well known Chebotarev problem on the existence of a continuum of minimal (standard) capacity among all continua which contain the points a_1, \dots, a_q . Chebotarev’s problem was solved in the 1930’s by G. Grötzsch, and independently by M. A. Lavrentiev. G. Grötzsch [13] carried out the proof of the uniqueness of the extremal continuum, following his ‘strip method’. Under the additional basic assumption

that the continuum in question is a union of analytic curves, M. A. Lavrentiev [19], [20] solved Chebotarev's problem applying a variation-geometrical approach.

In a series of articles from 1946 through 1951, G. M. Goluzin created his own method, different from the earlier approach introduced by M. Schiffer; namely, the so-called method of internal variations. Applying his method, G. M. Goluzin solved Chebotarev's problem in terms of quadratic differentials (see [17]).

Theorem of Goluzin ((see [17])). *Under conditions (10), the family of compacta $\mathfrak{K}_{\{\infty\},f}$ possesses a unique minimizing element F with respect to the measure δ_∞ (or, equivalently, to the field $\mathcal{V}^{-\delta_\infty} = 0$). The continuum F does not separate the plane; furthermore, F is a union of the closures of the critical trajectories of the quadratic differential $\frac{-B(z)}{A(z)}dz^2$, where $A(z)$ and $B(z)$ are monic polynomials, $A(z) = (z - a_1) \dots (z - a_q)$, $B(z)$ is of degree $q - 2$ and uniquely determined by the polynomial A . Green's function $g(z, \infty)$ of the domain $\overline{\mathbb{C}} \setminus F$ with pole at infinity is given by*

$$g(z, \infty) = \operatorname{Re} \int_a^z \sqrt{B(t)/A(t)} dt.$$

The information about zeros of the polynomial $B(z)$ results from conditions related to the connectivity of the union of the closures of the critical trajectories of the quadratic differentials $\frac{-B(z)}{A(z)}dz^2$. The explicit evaluation of $B(z)$ is a very hard problem studied, so far, only in the case $q = 3$. For details on Chebotarev's problem, and some other related problems in the theory of functions, as well, the reader may be referred to [16], [17], [35].

Note that Chebotarev's problem concerns a concrete function f (10). In contrast to it, in the eighties of the last century H. Stahl gave a positive answer to the general problem of existence of a minimizing element in the class $\mathfrak{K}_{\{\infty\},f}$ for an arbitrary function f in $\mathcal{A}(\{\infty\})$. Furthermore, Stahl described the extremal compactum F in terms of a quadratic differential. He established its symmetry property (in the absence of an external field) and using the symmetry, deduced that the classical diagonal Padé approximants converge in capacity to the function f in the domain $D := \overline{\mathbb{C}} \setminus F$. The latter is known as Stahl's domain; see [29]–[33].

We remind the reader of the definition of the classical diagonal Padé approximant R_n of a function $f \in \mathcal{H}(\{\infty\})$; that is $R_n = P_n/Q_n$, where

$$\deg P_n \leq n, \quad \deg Q_n \leq n, \quad Q_n \neq 0, \quad (11)$$

and

$$(Q_n f - P_n)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \rightarrow \infty. \quad (12)$$

Here, and in the sequel, we denote $\mathcal{H}(E)$ the class of functions holomorphic on E (i.e., in some neighborhood of E).

Theorem of Stahl . *Let $f \in \mathcal{A}(\{\infty\})$. Then there is a compactum $F \in \mathfrak{K}_{\{\infty\},f}$ such that*

1°. F is of minimal capacity; i.e.,

$$\text{cap } F = \min_{K \in \mathfrak{K}_{\{\infty\},f}} \text{cap } K.$$

2°. F is symmetric (in the field $\mathcal{V}^{-\delta_\infty} = 0$), and the complement $\overline{\mathbb{C}} \setminus F$ is connected.

3°. Let $R_n = P_n/Q_n$, $n = 1, 2, \dots$, be the classical diagonal Padé approximants of f . Then

$$\frac{1}{n} \delta_{Q_n} \xrightarrow{*} \mu_F^{\text{eq}} \quad \text{and} \quad R_n \xrightarrow{\text{cap}} f, \quad z \in \overline{\mathbb{C}} \setminus F, \quad \text{as } n \rightarrow \infty,$$

where μ_F^{eq} is the equilibrium measure of F . The convergence is characterized by the relation

$$|(f - R_n)(z)|^{1/n} \xrightarrow{\text{cap}} e^{-2g_F(z, \infty)}.$$

We note again that in the special case when $E = \{\infty\}$, the external field is absent (i.e. $\mathcal{V}^{-\delta_\infty} = 0$) and $f \in \mathcal{A}(\{\infty\})$, statement 1° of Stahl's Theorem yields, in essence, a positive answer to the existence of the δ_∞ -minimizing element in the class $\mathfrak{K}_{\{\infty\},f}$.

It turned out later that the implication $1^\circ \Rightarrow 2^\circ$ in Stahl's Theorem remains valid in quite more general situations. E. Rakhmanov and A. Martinez-Finkelshtein [21] (see also [26], [23], [15]) proved, using the variational method, that the μ -minimizing element F in $\mathfrak{K}_{E,f}$, which consists of a finite number of continua is necessarily an S -compactum in the external field $\mathcal{V}^{-\mu}$. In [21], the case of a discrete measure μ consisting of finitely many point masses was treated. In the present paper, we consider the case of a general measure μ . The variational method introduced by Rakhmanov and Martinez-Finkelshtein is crucial for the proof of the following theorem.

Theorem 1. *Let $E = \bigsqcup_{j=1}^m E_j \in \mathcal{E}_m$, $f \in \mathcal{A}(E)$, $\mu \in M(E)$, $\mu(E_j) > 0$, $j = 1, \dots, m$. Suppose that the μ -minimizing element F in the class $\mathfrak{K}_{E,f}$ consists of a finite number of continua. Then F is symmetric in the external field $\mathcal{V}^{-\mu}$ and $\overline{\mathbb{C}} \setminus F = \bigcup_{j=1}^m D_j$, where $D_j \supset E_j$. Moreover, the domains D_j and D_k , $j, k = 1, \dots, m$, either do not intersect one another, or coincide.*

Assume that K is the μ -minimizing compactum in $\mathfrak{K}_{E,f}$. It is worth noting that it is not difficult to find a compactum $F \subseteq K$ consisting of a finite number of continua and belonging to $\mathfrak{K}_{E,f}$. The particular case when all compacta E_j are single points $\{e_j\}$, $j = 1, \dots, m$, was considered in [6].

The implication $2^\circ \Rightarrow 3^\circ$ in Stahl's Theorem can be extended to the more general case of multipoint Padé approximants; for the definition see below.

Let $\tilde{E}_n = \{e_{n,1}, \dots, e_{n,2n+1}\}$ be a point set in the extended complex plane $\overline{\mathbb{C}}$ (each point preserves its multiplicity), and let

$$\omega_{\tilde{E}_n}(z) = \prod_{|e_j| \leq 1} (z - e_j) \prod_{|e_j| > 1} (1 - z/e_j)$$

be a spherically normalized polynomial associated with the set \tilde{E}_n . Suppose that $f \in \mathcal{H}(\tilde{E}_n)$.

The multipoint Padé approximant of order n of the function f (at the points in \tilde{E}_n) is the rational function $R_n = P_n/Q_n$ determined by (11) and by

$$\frac{fQ_n - P_n}{\omega_{\tilde{E}_n}}(z) \in \mathcal{H}(\tilde{E}_n \cap \mathbb{C}), \quad (13)$$

$$z^{n+1} \frac{fQ_n - P_n}{\omega_{\tilde{E}_n}}(z) \in \mathcal{H}(\tilde{E}_n \cap \{\infty\}) \quad (14)$$

(if $\tilde{E}_n \cap \mathbb{C} = \emptyset$ or $\tilde{E}_n \cap \{\infty\} = \emptyset$, then condition (13) or (14), respectively, fails).

The special case $e_{1,n} = \dots = e_{2n+1,n}$ coincides with the concept of classical Padé approximant of f at infinity, if $e_{1,n} = \infty$ (or at zero, if $e_{1,n} = 0$).

Everywhere below we assume that $f \in \mathcal{H}(E)$. Furthermore, we assume that the sets $\tilde{E}_n = \{e_{n,1}, \dots, e_{n,2n+1}\}$, $n = 1, 2, \dots$, of the nodes of interpolation are located on E , and their limit distribution is described by

$$\frac{1}{2n} \delta_{\tilde{E}_n} \xrightarrow{*} \mu, \quad n \rightarrow \infty.$$

For tables of interpolation nodes, satisfying the above conditions we adopt the notation (E, μ) .

Let $F \in \mathcal{F}$, and Ω be a neighborhood of the compactum F . We denote $\mathcal{H}_0(\Omega \setminus F)$ the set of all functions f which are holomorphic in $\Omega \setminus F$ (or piecewise holomorphic, if $\Omega \setminus F$ is not connected), and possess continuous limit values from both sides on the arcs of $F_0 \setminus B$, where $B = B(f)$ is a certain compactum of zero capacity, wherein the jump χ_f of f does not have zeros on $F_0 \setminus B$.

A. A. Gonchar and E. A. Rakhmanov proved [10] a fundamental theorem (see also [12]) on the asymptotic behavior of the zeros of polynomials, satisfying non-Hermitian orthogonality conditions on a compactum which possesses the symmetry property. The notion of symmetry of a compactum $F \in \mathcal{F}$ was introduced in the present paper only for the case of harmonic fields of the form $\psi(z) = \mathcal{V}^{-\mu}(z)$ with $S(\mu) \cap F = \emptyset$. For this reason, we formulate Gonchar–Rakhmanov’s theorem only for harmonic fields of such type.

Theorem of Gonchar–Rakhmanov . *Let $F \subset \mathbb{C}$ be a compactum of positive capacity, Ω a neighborhood of F , and $\mu \in M(\overline{\mathbb{C}} \setminus \Omega)$. Assume that the sequence of functions Ψ_n and the function f satisfy the following conditions:*

1°. Ψ_n are holomorphic in Ω , $n = 1, 2, \dots$, and

$$\psi_n(z) = \frac{1}{2n} \log \frac{1}{|\Psi_n(z)|} \Rightarrow \psi(z) = \mathcal{V}^{-\mu}(z), \quad z \in \Omega.$$

2°. F is symmetric in the external field $\mathcal{V}^{-\mu}$.

3°. The complement $\overline{\mathbb{C}} \setminus F$ is connected.

4°. $f \in \mathcal{H}_0(\Omega \setminus F)$.

If the polynomials Q_n , $\deg Q_n \leq n$ ($Q_n \not\equiv 0$) satisfy the orthogonality conditions

$$\oint_F Q_n(t) \Psi_n(t) f(t) t^\nu dt = 0, \quad \nu = 0, 1, \dots, n-1 \quad (n = 1, 2, \dots), \quad (15)$$

then the following statements are true:

- (i) $\frac{1}{n} \delta_{Q_n} \xrightarrow{*} \tilde{\mu}_F$ as $n \rightarrow \infty$, where $\tilde{\mu}_F$ stands for the balayage of μ onto F .
- (ii) if Q_n are spherically normalized, then

$$\left| \oint_F \frac{Q_n^2(t) \Psi_n(t) f(t) dt}{t - z} \right|^{1/n} \xrightarrow{\text{cap}} e^{-2w_F^\mu}, \quad z \in \overline{\mathbb{C}} \setminus F, \quad (16)$$

where w_F^μ is the constant from (6).

As a corollary, Gonchar and Rakhmanov deduced the following result.

Corollary of Gonchar–Rakhmanov theorem . *Let F be a compactum symmetric in the field $\mathcal{V}^{-\mu}$, where $\mu \in M(E)$ and $E \cap F = \emptyset$. Suppose that the complement $\overline{\mathbb{C}} \setminus F$ is connected and let $\{R_n\}_{n=1}^{\infty}$ be the sequence of multipoint Padé approximants P_n/Q_n of the function $f \in \mathcal{H}_0(\overline{\mathbb{C}} \setminus F)$, associated with the (E, μ) -table of nodes of interpolation. Then*

- (i) $\frac{1}{n} \delta_{Q_n} \xrightarrow{*} \tilde{\mu}_F$, where $\tilde{\mu}_F$ means the balayage of μ onto F .
- (ii) $R_n(z) \xrightarrow{\text{cap}} f(z)$, $z \in \overline{\mathbb{C}} \setminus F$. The rate of convergence is given by

$$|(f - R_n)(z)|^{1/n} \xrightarrow{\text{cap}} e^{-2G_F^\mu(z)}.$$

Subsequently, it was proved [5] that under the condition $\psi(z) = \mathcal{V}^{-\mu}(z)$, the requirement 3° in Gonchar–Rakhmanov’s theorem concerning connectivity of the complement $\overline{\mathbb{C}} \setminus F$, can be weakened to the condition that each connected component of the complement $\overline{\mathbb{C}} \setminus F$ contains a nonempty inner boundary arc. For details, the reader is referred to [5] and [6].

Theorem 1 and the Corollary of Gonchar–Rakhmanov’s theorem are focused on the fact that the positive solution of the μ -minimization problem in the set $\mathfrak{K}_{E,f}$ is of crucial importance in the theory of rational approximations. Below we list some cases when a positive answer has been already achieved. In each of these cases, the following line of reasoning is used.

Let F_n , $n = 1, 2, \dots$ be a sequence of compacta in $\mathfrak{K}_{E,f}$, each of them being a union of a finite number (independent on n) of continua, and such that

$$\lim_{n \rightarrow \infty} \text{cap}_\mu F_n = \inf_{K \in \mathfrak{K}_{E,f}} \text{cap}_\mu K.$$

Suppose that there is a mapping $T: \mathbb{C} \rightarrow \mathbb{C}$ with the property

$$T(F_n) \in \mathfrak{K}_{E,f}, \quad \text{cap}_\mu T(F_n) \leq \text{cap}_\mu F_n \quad \text{and} \quad T(F_n) \subset \mathbb{C} \setminus \Omega,$$

where Ω is some neighborhood of the compactum E which is independent on n . Then, extracting from the sequence $\{T(F_n)\}_{n=1}^{\infty}$ a subsequence which converges to some compactum $F = \lim_{n \in \Lambda} T(F_n)$ in the Hausdorff metric, and applying equality (3), we obtain the equality

$$\text{cap}_\mu F = \lim_{n \in \Lambda} \text{cap}_\mu T(F_n) = \inf_{K \in \mathfrak{K}_{E,f}} \text{cap}_\mu K.$$

It is not difficult to check that $F \in \mathfrak{K}_{E,f}$. Thus, the compact F solves the μ -minimization problem in the set $\mathfrak{K}_{E,f}$.

While proving the first statement of his theorem, H. Stahl showed that if all singularities of $f \in \mathcal{A}(\{\infty\})$ are located in the disk $U_R = \{|z| \leq R\}$, then as the mapping T one can take the radial projection T_R onto the disk U_R ; that is

$$\begin{aligned} T_R z &= z, & \text{for } |z| \leq R, \\ T_R z &= z \frac{R}{|z|}, & \text{for } |z| > R. \end{aligned}$$

The case of the so-called two-point Padé approximants when $E = (\{0\} \cup \{\infty\}) \in \mathcal{E}_2$, $f \in \mathcal{A}(E)$ and the measure $\mu = \frac{\delta_0 + \delta_\infty}{2}$ was considered in [8]. It was shown that as the mapping T one can take the radial projection onto the annulus $U_{r,R} = \{r \leq |z| \leq R\}$ ($0 < r < R < \infty$), which contains the set A_f of the singularities of f . The mapping $T = T_{r,R}$ is defined as follows

$$\begin{aligned} T_{r,R} z &= z, & \text{for } z \in U_{r,R}, \\ T_{r,R} z &= z \frac{r}{|z|}, & \text{for } |z| < r, \\ T_{r,R} z &= z \frac{R}{|z|}, & \text{for } |z| > R. \end{aligned}$$

The proposition used in [8] on the decay of $\text{cap}_\mu K$ under radial projection onto the annulus $U_{r,R}$ is no longer true if $\mu \neq \frac{\delta_0 + \delta_\infty}{2}$. Nevertheless, it was shown in [6] that the posed problem is also solvable, if $E = \bigsqcup_{j=1}^m \{e_j\} \in \mathcal{E}_m$, $f \in \mathcal{A}(E)$, $\mu = \sum_{j=1}^m p_j \delta_{e_j} \in M(E)$ ($p_j \geq 0$, $\sum p_j = 1$).

The solution of the problem is also positive in the case when $E \in \mathcal{E}_1$, $E \subseteq \{|z| \geq 1\}$, $\mu \in M(E)$ and $f \in \mathcal{A}(E)$ such that $A_f \subset U_R = \{|z| \leq R\}$, $R < 1$. More exactly, L. Baratchart, H. Stahl and M. Yattselev established [3] that the mapping T_R onto the disk U_R , as defined above possesses the desired properties. To be precise, we present their result in [3] in the form that is suitable for the purpose of the current paper.

Theorem of Baratchart–Stahl–Yattselev . *Let $E \in \mathcal{E}_1$, $E \subseteq \{|z| \geq 1\}$, $\mu \in M(E)$, $f \in \mathcal{A}(E)$ and $A_f \subset U_R = \{|z| \leq R\}$, where $R < 1$. Then there is a μ -minimizing element F in the family $\mathfrak{K}_{E,f}$ and $F \subseteq U_R$. This μ -minimizing element F is symmetric in the external field $\mathcal{V}^{-\mu}$, and F is a union of closures of the critical trajectories of a quadratic differential.*

Note that in a general case the minimization problem in $\mathfrak{K}_{E,f}$ with respect to the measure (with no additional assumptions) may not be solvable. This may happen in the case of $E \in \mathcal{E}_m$, $\mu \in M(E)$ and $f \in \mathcal{A}(E)$, selected in a special way. In fact, it is easy to construct a continuum $E \in \mathcal{E}_1$, a measure $\mu \in M(E)$ and a function $f \in \mathcal{A}(E)$ such that the inequality

$$\text{cap}_\mu F > \inf_{K \in \mathfrak{K}_{E,f}} \text{cap}_\mu K. \quad (17)$$

holds for every $F \in \mathfrak{K}_{E,f}$.

In the present paper, we show that such an example applies not only to the case when $E \in \mathcal{E}_1$ and $\mu \in M(E)$ selected in a special way, but also to the concrete simple continuum $E = [a, b]$, for some sufficiently wide class of natural measures supported on $[a, b]$, and appropriate functions $f \in \mathcal{A}(E)$.

More precisely let $E = [a, b] \subset \mathbb{C}$. Denote $\widetilde{M}(E)$ the subclass of all measures μ from class $M(E)$, such that the logarithmic potential of μ , $V^\mu(z) := -\int \log |z - t| d\mu(t)$, is continuous at some inner point $z_0 = z_0(\mu)$ of the closed interval E , and the condition

$$\lim_{t \rightarrow 0} \mu(\{|z - z_0| \leq t\}) = 0 \quad (18)$$

is satisfied. We can easily see that both the equilibrium (Chebyshev) measure for E , and the normalized Lebesgue measure, possess these properties, and thus they are both contained in $\widetilde{M}(E)$.

Theorem 2. *Let $E = [a, b] \subset \mathbb{C}$ be a closed interval, $\mu \in \widetilde{M}(E)$. There exists a function $f \in \mathcal{A}(E)$ such that the μ -minimization problem (9) in the family of compacta $\mathfrak{K}_{E,f}$ is not solvable.*

To conclude this section we remark that in view of Theorem 1 the μ -minimizing compactum from the set $\mathfrak{K}_{E,f}$ with respect to the measure μ possesses an S -property. The existence of a compactum with the S -property is crucial not only in the study of distribution of zeros of orthogonal polynomials, but also in the derivation of the formulas of strong asymptotics, on the basis of the matrix Riemann–Hilbert method, and other methods as well (see [24], [34], [4], [1], [11], [9], [2], [23], [27], [14]).

The results of the present paper were partly announced in [7].

3 Proof of Theorem 1

Under the conditions of Theorem 1, the μ -minimizing element F in the class $\mathfrak{K}_{E,f}$, where $E = \bigsqcup_{j=1}^m E_j \in \mathcal{E}_m$, $\mu \in M(E)$, consists of a finite number of continua. Let $A_f = \bigcup_{j=1}^m A_{f_j} = \{a_1, \dots, a_p\}$ be the set of all singularities of $f \in \mathcal{A}(E)$.

Fix $w \in \mathbb{C} \setminus F$ and write $h_w(z) = \frac{A_p(z)}{z-w}$, where $A_p(z) = \prod_{l=1}^p (z - a_l)$. Applying standard arguments, we define the mapping (“variation”) $z \mapsto z_t^h = z + th_w(z)$ (see [25], [21], [22]), where $t \in \mathbb{C}$ is a complex-valued parameter.

If t is small enough, say $0 < |t| \leq \varepsilon_0$, then the mapping is univalent in a neighborhood of F . Furthermore, $F \mapsto \{z_t^h, z \in F\} = F_t^h$, the set A_f of the singularities of f remains stable under the mapping $z \mapsto z_t^h$, and the measure $\nu \in M(F)$ maps to the measure $\nu_t^h \in M(F_t^h)$ where $\nu_t^h(B_t^h) = \nu(B)$ for every $B_t^h \subset F_t^h$. Thus, $d\nu_t^h(z_t^h) = d\nu(z)$. Taking into account the definition of the class $\mathcal{A}(E)$, the stability of the set A_f under the mapping $z \mapsto z_t^h$ and the condition that F consists of a finite number of continua, we can conclude that $F_t^h \in \mathfrak{K}_{E,f}$.

We remind the definition of the energy $I_\mu(\nu)$ of a measure $\nu \in M(F)$ in the presence of the external field $\mathcal{V}^{-\mu}$ (i.e., the μ -weighted energy):

$$I_\mu(\nu) := - \iint \log |z - \zeta| d\nu(z) d\nu(\zeta) + 2 \int \mathcal{V}^{-\mu}(z) d\nu(z). \quad (19)$$

In accordance with this definition, and that of the measure ν_t^h we have

$$\begin{aligned} I_\mu(\nu_t^h) &= - \iint \log |z - \zeta| d\nu_t^h(z) d\nu_t^h(\zeta) + 2 \int \mathcal{V}^{-\mu}(z) d\nu_t^h(z) \\ &= - \iint \log |z_t^h - \zeta_t^h| d\nu(z) d\nu(\zeta) + 2 \int \mathcal{V}^{-\mu}(z_t^h) d\nu(z). \end{aligned}$$

Subtracting equality (19) from the latter, we get a formula which describes the increment of the energy of the measure ν in the field $\mathcal{V}^{-\mu}$ under the mapping $z \mapsto z_t^h$:

$$\begin{aligned} I_\mu(\nu_t^h) - I_\mu(\nu) &= - \iint \log \left| \frac{z_t^h - \zeta_t^h}{z - \zeta} \right| d\nu(z) d\nu(\zeta) + 2 \iint \log \left| \frac{z_t^h - \zeta}{z - \zeta} \right| d\mu(\zeta) d\nu(z) \\ &= \operatorname{Re} \left\{ - \iint \log \left(1 + \frac{t(h_w(z) - h_w(\zeta))}{z - \zeta} \right) d\nu(z) d\nu(\zeta) \right. \\ &\quad + 2 \iint \log \left(1 + \frac{th_w(z)}{z - \zeta} \right) d\mu(\zeta) d\nu(z) \left. \right\} = \operatorname{Re} \left\{ -t \iint \frac{(h_w(z) - h_w(\zeta))}{z - \zeta} d\nu(z) d\nu(\zeta) \right. \\ &\quad \left. + 2t \iint \frac{h_w(z)}{z - \zeta} d\mu(\zeta) d\nu(z) \right\} + O(t^2) = \operatorname{Re} t H_{w,\mu}(\nu) + O(t^2), \quad (20) \end{aligned}$$

where

$$H_{w,\mu}(\nu) = - \iint \frac{h_w(z) - h_w(\zeta)}{z - \zeta} d\nu(\zeta) d\nu(z) + 2 \iint \frac{h_w(z)}{z - \zeta} d\mu(\zeta) d\nu(z). \quad (21)$$

Applying equality (20) to the measures $\tilde{\mu}_F$ and $\sigma_{t,h}$, where $\sigma_{t,h}$, in the class $M(F)$ is such that $(\sigma_{t,h})_t^h = \tilde{\mu}_{F_t^h}$, we obtain two equalities

$$I_\mu((\tilde{\mu}_F)_t^h) - I_\mu(\tilde{\mu}_F) = \operatorname{Re} t H_{w,\mu}(\tilde{\mu}_F) + O(t^2) = O(t), \quad (22)$$

$$I_\mu(\tilde{\mu}_{F_t^h}) - I_\mu(\sigma_{t,h}) = \operatorname{Re} t H_{w,\mu}(\sigma_{t,h}) + O(t^2) = O(t). \quad (23)$$

It is well known [28] that for each compactum K whose intersection with $S(\mu)$ is empty, the following equalities are valid:

$$I_\mu(\tilde{\mu}_K) = \inf_{\nu \in M(K)} I_\mu(\nu) \quad (24)$$

and

$$\operatorname{cap}_\mu K = e^{-I_\mu(\tilde{\mu}_K)}. \quad (25)$$

Therefore,

$$I_\mu(\sigma_{t,h}) \geq I_\mu(\tilde{\mu}_F) \quad \text{and} \quad I_\mu((\tilde{\mu}_F)_t^h) \geq I_\mu(\tilde{\mu}_{F_t^h}). \quad (26)$$

Using (23), the second inequality in (26) and (22), we arrive at

$$I_\mu(\sigma_{t,h}) = I_\mu(\tilde{\mu}_{F_t^h}) + O(t) \leq I_\mu((\tilde{\mu}_F)_t^h) + O(t) = I_\mu(\tilde{\mu}_F) + O(t),$$

Combining this inequality and the first inequality in (26) we obtain, by letting $t \rightarrow 0$

$$I_\mu(\sigma_{t,h}) \rightarrow I_\mu(\tilde{\mu}_F). \quad (27)$$

It is well known that $\tilde{\mu}_K$ is the unique measure in $M(K)$ for which (24) holds. Taking into account this observation, the principle of descent [18, chapter I, §3, Theorem 1.3] and (27), we deduce that $\sigma_{t,h}$ converges weakly to the measure $\tilde{\mu}_F$ as $t \rightarrow 0$. This yields

$$\lim_{t \rightarrow 0} H_{w,\mu}(\sigma_{t,h}) = H_{w,\mu}(\tilde{\mu}_F). \quad (28)$$

As noticed previously, $F_t^h \in \mathfrak{K}_{E,f}$ for all t small enough. Under the conditions of Theorem 1, the compactum F is the μ -minimizing element in $\mathfrak{K}_{E,f}$. Thus $\operatorname{cap}_\mu F \leq \operatorname{cap}_\mu F_t^h$, which thanks to (25), is equivalent to the inequality $I_\mu(\tilde{\mu}_F) \geq I_\mu(\tilde{\mu}_{F_t^h})$. Using this inequality, the first inequality in (26) and equality (23), we obtain

$$0 \geq I_\mu(\tilde{\mu}_{F_t^h}) - I_\mu(\tilde{\mu}_F) \geq I_\mu(\tilde{\mu}_{F_t^h}) - I_\mu(\sigma_{t,h}) = \operatorname{Re} t H_{w,\mu}(\sigma_{t,h}) + O(t^2),$$

Since the parameter $t \rightarrow 0$ is arbitrary, the latter inequality is possible only if $\lim_{t \rightarrow 0} H_{w,\mu}(\sigma_{t,h}) = 0$. From here and from (28) we obtain

$$H_{w,\mu}(\tilde{\mu}_F) = 0. \quad (29)$$

Applying the definition (21) to $\nu = \tilde{\mu}_F$ on the left hand side of equality (29), as well as the explicit representation of the function $h_w(z) = \frac{A_p(z)}{z-w}$, we rewrite (29) as follows:

$$\begin{aligned} & - \iint \left(\frac{A_p(z)}{(z-w)(z-\zeta)} - \frac{A_p(\zeta)}{(\zeta-w)(z-\zeta)} \right) d\tilde{\mu}_F(z) d\tilde{\mu}_F(\zeta) + \\ & + 2 \iint \frac{A_p(z)}{(z-w)(z-\zeta)} d\mu(\zeta) d\tilde{\mu}_F(z) \equiv 0, \quad w \in \mathbb{C} \setminus (E \cup F). \end{aligned} \quad (30)$$

We note that

$$A_p(z) - A_p(w) = (z-w)A_{p,1}(z, w),$$

where $A_{p,1}(z, w)$ is a polynomial of degree $p-1$ in each variable z, w . Therefore,

$$\frac{A_p(z)}{(z-w)(z-\zeta)} = \frac{A_p(w)}{(z-w)(z-\zeta)} + \frac{A_{p,1}(z, w)}{z-\zeta} = \frac{A_p(w)}{\zeta-w} \left(\frac{1}{z-\zeta} - \frac{1}{z-w} \right) + \frac{A_{p,1}(z, w)}{z-\zeta}. \quad (31)$$

We also note that the expression

$$A_p(z)(w-\zeta) + A_p(w)(\zeta-z) + A_p(\zeta)(z-w)$$

is a polynomial of degree p in each variable z, w, ζ , with zeros at $z = w$, $z = \zeta$, $w = \zeta$. Thus

$$\frac{A_p(z)}{(z-w)(\zeta-z)} + \frac{A_p(w)}{(w-\zeta)(z-w)} + \frac{A_p(\zeta)}{(\zeta-z)(w-\zeta)} = A_{p,2}(z, w, \zeta),$$

where $A_{p,2}(z, w, \zeta)$ is a polynomial of degree $p-2$ in each z, w, ζ . Consequently,

$$\frac{A_p(z)}{(z-w)(z-\zeta)} - \frac{A_p(\zeta)}{(\zeta-w)(z-\zeta)} = -A_{p,2}(z, w, \zeta) - \frac{A_p(w)}{(z-w)(\zeta-w)}. \quad (32)$$

For $w \in \overline{\mathbb{C}} \setminus (E \cup F)$ we obtain, by substitution of (31) and (32) into (30),

$$0 = \iint \left(A_{p,2}(w, z, \zeta) + \frac{A_p(w)}{(z-w)(\zeta-w)} \right) d\tilde{\mu}_F(z) d\tilde{\mu}_F(\zeta) +$$

$$\begin{aligned}
& 2 \iint \left(\frac{A_p(w)}{\zeta - w} \left(\frac{1}{z - \zeta} - \frac{1}{z - w} \right) + \frac{A_{p,1}(w, z)}{z - \zeta} \right) d\mu(\zeta) d\tilde{\mu}_F(z) \\
&= A_{p,3}(w) + A_p(w) \varphi_{\tilde{\mu}_F}(w)^2 + 2A_p(w) \int \frac{\varphi_{\tilde{\mu}_F}(\zeta)}{\zeta - w} d\mu(\zeta) - 2A_p(w) \varphi_{\tilde{\mu}_F}(w) \varphi_\mu(w),
\end{aligned}$$

where

$$A_{p,3}(w) = \iint A_{p,2}(w, z, \zeta) d\tilde{\mu}_F(z) d\tilde{\mu}_F(\zeta) + 2 \iint \frac{A_{p,1}(w, z)}{z - \zeta} d\mu(\zeta) d\tilde{\mu}_F(z)$$

is a polynomial of degree not greater than $p - 1$. The integrals

$$\varphi_{\tilde{\mu}_F}(w) = \int \frac{d\tilde{\mu}_F(z)}{z - w} \quad \text{and} \quad \varphi_\mu(w) = \int \frac{d\mu(z)}{z - w}, \quad w \in \overline{\mathbb{C}} \setminus (E \cup F).$$

represent the Cauchy transformations of the measures $\tilde{\mu}_F$ and μ , respectively.

Rewrite the equality which we obtained as follows:

$$(\varphi_{\tilde{\mu}_F}(w) - \varphi_\mu(w))^2 = \varphi_\mu(w)^2 - \frac{A_{p,3}(w)}{A_p(w)} - 2 \int \frac{\varphi_{\tilde{\mu}_F}(\zeta)}{\zeta - w} d\mu(\zeta). \quad (33)$$

Denote the right hand side of (33) by $R(w)$. The function $R(w)$ is meromorphic on F , and has poles only at the zeros of the polynomial $A_p(w)$.

By virtue of (33), we derive for the multi-valued complex potential $\mathfrak{V}^{\mu - \tilde{\mu}_F}(z)$ of the charge $\mu - \tilde{\mu}_F$

$$\begin{aligned}
\mathfrak{V}^{\mu - \tilde{\mu}_F}(z) &:= \int \log(z - \zeta) d(\tilde{\mu}_F - \mu)(\zeta) = \\
&= \int \left(\int^z \frac{1}{w - \zeta} dw \right) d(\tilde{\mu}_F - \mu)(\zeta) = \int^z (\varphi_\mu(w) - \varphi_{\tilde{\mu}_F}(w)) dw = \int^z \sqrt{R(w)} dw.
\end{aligned}$$

Hence, for the corresponding real parts we have

$$\mathcal{V}^{\mu - \tilde{\mu}_F}(z) = \operatorname{Re} \int^z \sqrt{R(w)} dw. \quad (34)$$

It follows from (6) that $\mathcal{V}^{\mu - \tilde{\mu}_F}(z)$ is, up to a constant w_F^μ , identical to $G_F^\mu(z)$.

Hence, by (34) and in view of the fact that $G_F^\mu = 0$ on F ,

$$G_F^\mu(z) = \operatorname{Re} \int_a^z \sqrt{R(w)} dw, \quad \text{where } a \in F.$$

Since $F = \overline{\{z : G_F^\mu(z) = 0\}}$, we have

$$F = \overline{\left\{z : \operatorname{Re} \int_a^z \sqrt{R(w)} dw = 0\right\}}.$$

Thus, F consists of a finite number of analytic arcs, the endpoints of which belong to the union of the set $\{a_1, \dots, a_p\}$ and the set of zeros of the function R .

Below we prove that the compactum F is symmetric in the external field $\mathcal{V}^{-\mu}$ (or, in the other words, we establish equality (8)). Let $\zeta \in F_0$. Denote $\nabla_\pm \mathcal{V}^{\mu-\tilde{\mu}_F}$ the limit values of the gradient of the potential $\mathcal{V}^{\mu-\tilde{\mu}_F}$. Since $\mathcal{V}^{\mu-\tilde{\mu}_F} \equiv \text{const}$ for $z \in F$, then F turns out to be a level curve of $\mathcal{V}^{\mu-\tilde{\mu}_F}$ (i.e., an equipotential curve), which implies

$$\left| \frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_\pm}(\zeta) \right| = \left| \nabla_\pm \mathcal{V}^{\mu-\tilde{\mu}_F}(\zeta) \right|, \quad \zeta \in F_0.$$

On the other hand, $\mathcal{V}^{\mu-\tilde{\mu}_F} = \operatorname{Re} \mathfrak{V}^{\mu-\tilde{\mu}_F}$ and $\frac{d\mathfrak{V}^{\mu-\tilde{\mu}_F}}{d\zeta}(\zeta) = \sqrt{R(\zeta)}$. Hence,

$$|\nabla \mathcal{V}^{\mu-\tilde{\mu}_F}(\zeta)| = \left| \frac{d\mathfrak{V}^{\mu-\tilde{\mu}_F}}{d\zeta}(\zeta) \right| = |\sqrt{R(\zeta)}|, \quad \zeta \in \mathbb{C} \setminus F.$$

Therefore, $|\nabla_+ \mathcal{V}^{\mu-\tilde{\mu}_F}(\zeta)| = |\nabla_- \mathcal{V}^{\mu-\tilde{\mu}_F}(\zeta)|$ which in turn yields that

$$\left| \frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_+}(\zeta) \right| = \left| \frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_-}(\zeta) \right|, \quad \zeta \in F_0.$$

Since $G_F^\mu(z) = 0$ for $z \in F$ and $G_F^\mu(z) > 0$ for $z \in \mathbb{C} \setminus F$, we have $\frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_\pm}(\zeta) > 0$, $\zeta \in F_0$. From here we derive the equality

$$\frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_+}(\zeta) = \frac{\partial \mathcal{V}^{\mu-\tilde{\mu}_F}}{\partial n_-}(\zeta), \quad \zeta \in F_0,$$

which coincides with (8).

Now, let the connected component G of the complement of F be such that $G \cap E = G \cap S(\mu) = \emptyset$. Then the function on the left hand side of the equality (34) is harmonic in the domain G and equals a constant on $\partial G \subset F$. Thus this function is identically constant on G . The last statement

contradicts the right hand side of (34). Consequently, each connected component of F has necessarily a nonempty intersection with E . Since $E = \bigsqcup_{j=1}^m E_j \in \mathcal{E}_m$, then each connected component contains entirely one or several continua E_j , $j = 1, \dots, m$.

The proof of Theorem 1 is complete.

4 Proof of Theorem 2

Without loss of generality, we may assume that $E = [a, b]$ is located on the real axis, and $a \leq -1$, $b = 1$, $z_0 = 0$. Given $k \in \mathbb{N}$, the linear function $z \mapsto kz$ maps E onto the closed interval $E_k = [ka, k]$, where $ka \leq -k$.

By this, the measure $\mu \in \widetilde{M}(E)$ transfers to a measure $\mu_k \in M(E_k)$, where $\mu_k(B) = \mu(\{z : kz \in B\})$; therefore, $d\mu_k(kz) = d\mu(z)$.

Consider the multi-valued analytic function

$$f(z) = \sqrt{(z - a_1)(z - a_2)(z - a_3)^{-1}(z - a_4)^{-1}},$$

where

$$a_1 = \frac{-2 + 3i}{16}, \quad a_2 = \frac{2 + 3i}{16}, \quad a_3 = \frac{-2 - i}{16}, \quad a_4 = \frac{2 - i}{16}.$$

Since for the branch points of f we have $\operatorname{Im} a_1, \operatorname{Im} a_2 > 0$ and $\operatorname{Im} a_3, \operatorname{Im} a_4 < 0$, we can take a holomorphic branch f^* of f on \mathbb{R} , determined by the condition $f^*(\infty) = 1$. Clearly, $f^* \in \mathcal{A}(E_k)$ for every $k = 1, 2, \dots$.

Denote L the union of the diagonals $[a_1, a_4]$ and $[a_2, a_3]$ of the square with vertices at the points a_1, a_2, a_3, a_4 . We keep the notation f_L for the holomorphic branch of f in the domain $\overline{\mathbb{C}} \setminus L$ determined by the condition $f_L(\infty) = 1$.

We easily see that the proof of Theorem 2 will be completed, if we show that there exists a number $k \in \mathbb{N}$ such that for all $F \in \mathfrak{K}_{E_k, f^*}$ the inequality

$$\operatorname{cap}_{\mu_k} F > \inf_{K \in \mathfrak{K}_{E_k, f^*}} \operatorname{cap}_{\mu_k} K \quad (35)$$

holds. Furthermore, we show that inequality (35) is true for all sufficiently large k and all $F \in \mathfrak{K}_{E_k, f^*}$.

Let

$$\psi_k(z) := V^{-\mu}(k^{-1}z) - V^{-\mu}(0).$$

Because of the conditions imposed on the measure μ , the potential $V^\mu(z)$ is continuous at the point $z = 0$. This implies the uniform convergence of the sequence of functions ψ_k on each compact set $K \subset \mathbb{C}$, in particular, on the disk $D = \{|z| \leq 1\}$, as well:

$$\psi_k(z) \Rightarrow 0, \quad z \in D, \quad k \rightarrow \infty. \quad (36)$$

We observe that the spherically normalized potential $\mathcal{V}^{\mu_k}(z)$, given by the equality (1), differs from the standard potential $V^{\mu_k}(z)$ by a constant. From here and from $d\mu_k(k\tau) = d\mu(\tau)$, we obtain

$$\begin{aligned} \mathcal{V}^{-\mu_k}(z) &= V^{-\mu_k}(z) + C_k = \int \log |z - t| d\mu_k(t) + C_k \\ &= \int \log |k(k^{-1}z - \tau)| d\mu(\tau) + C_k = \log k + C_k + V^{-\mu}(0) + \psi_k(z). \end{aligned}$$

Now, considering (2), we conclude that the inequality (35) is equivalent to the inequality

$$\text{cap}_{\psi_k}(F) > \inf_{K \in \mathfrak{R}_{E_k, f^*}} \text{cap}_{\psi_k}(K). \quad (37)$$

Denoting $\rho_k = \exp\{\max_{z \in D} |\psi_k(z)|\}$, we get for $\rho_k \geq 1$ the estimate

$$\rho_k^{-1} \leq e^{\psi_k(z)} \leq \rho_k, \quad z \in D.$$

Furthermore, by (36), $\lim_{k \rightarrow \infty} \rho_k = 1$. From here and from (2), after assuming that $K \subseteq D$, we deduce the estimates of the capacity $\text{cap}_{\psi_k} K$ (from below and above) in terms of the standard capacity $\text{cap} K$, namely

$$\rho_k^{-2} \text{cap} K \leq \text{cap}_{\psi_k}(K) \leq \rho_k^2 \text{cap} K. \quad (38)$$

Assume that the assertion that the inequality (37) holds for all k starting from a number k_0 and for all $F \in \mathfrak{R}_{E_k, f^*}$, is false. Then there is an increasing sequence $\{k_j\}_{j=1}^\infty$ of natural numbers and compacta $F_{k_j} \in \mathfrak{R}_{E_{k_j}, f^*}$ such that for all $j = 1, 2, \dots$

$$\text{cap}_{\psi_{k_j}}(F_{k_j}) = \inf_{K \in \mathfrak{R}_{E_{k_j}, f^*}} \text{cap}_{\psi_{k_j}}(K). \quad (39)$$

Without loss of generality, we may suppose that each compactum F_{k_j} contains at most two continua, and each of these continua contains a pair of points from the set $\{a_1, a_2, a_3, a_4\}$, see [3, Lemma 15].

We first establish the inclusion $F_{k_j} \subset D$ for all sufficiently large j . For this purpose, we observe that $K^* = [a_1, a_2] \cup [a_3, a_4] \in \mathfrak{K}_{E_{k_j}, f^*}$ and that K^* is contained in the disk of radius $2^{-3}\sqrt{2}$ centered at $a_0 = \sum_{l=1}^4 a_l/4 = 2^{-4}i$. Therefore, after taking into account the second inequality in (38), and the equality $\lim_{k \rightarrow \infty} \rho_k = 1$, for j large enough, say $j \geq j_0$, we derive the estimate

$$\inf_{K \in \mathfrak{K}_{E_{k_j}, f}} \text{cap}_{\psi_{k_j}}(K) \leq \text{cap}_{\psi_{k_j}}(K^*) \leq \rho_{k_j}^2 \text{cap} K^* \leq \rho_{k_j}^2 2^{-3}\sqrt{2} < 0.18. \quad (40)$$

On the other hand, if the compactum $F_{k_j} \in \mathfrak{K}_{E_{k_j}, f^*}$ (consisting of at most two continua, each of the latter containing two points from the set $\{a_1, a_2, a_3, a_4\}$), is not lying in the disk D , then F_{k_j} contains some continuum $F_{k_j}^*$ located in D and combining some point on the circle $\{|z| = 1\} = \partial D$ with some of the points a_1, a_2, a_3, a_4 . The distance from each of these points to ∂D is not smaller than $1 - 2^{-4}\sqrt{13}$. Therefore,

$$\text{cap} F_{k_j} \geq \text{cap} F_{k_j}^* \geq 2^{-2}(1 - 2^{-4}\sqrt{13}) > 0.19.$$

Hence, by the first inequality in (38), for sufficiently large j , say $j \geq j_1$, we have

$$\text{cap}_{\psi_{k_j}}(F_{k_j}) \geq \text{cap} F_{k_j} \rho_{k_j}^{-2} > 0.19 \rho_{k_j}^{-2} > 0.18.$$

The latter inequality and (40) yield for every $j \geq \max\{j_0, j_1\}$

$$\text{cap}_{\psi_{k_j}}(F_{k_j}) > \inf_{K \in \mathfrak{K}_{E_{k_j}, f^*}} \text{cap}_{\psi_{k_j}}(K),$$

which contradicts (39). Consequently, $F_{k_j} \subset D$ for all $j \geq \max\{j_0, j_1\}$.

From the inclusion $F_{k_j} \subset D$ and from the relation $F_{k_j} \cap E_{k_j} = \emptyset$ we obtain, in particular, that for $j \geq \max\{j_0, j_1\}$ the set F_{k_j} consists exactly of two continua $F_{k_j}^+$ and $F_{k_j}^-$; the former lies in $D \cap \{\text{Im } z > 0\}$ and connects a_1 with a_2 , the latter is located in $D \cap \{\text{Im } z < 0\}$ and connects a_3 with a_4 .

Recall the definition of the distance between two compacta $K_1, K_2 \subset \overline{\mathbb{C}}$ in the spherical Hausdorff metric, that is:

$$\rho_H(K_1, K_2) = \inf\{\delta : K_1 \subset K_2^\delta \quad \text{and} \quad K_2 \subset K_1^\delta\}, \quad (41)$$

where

$$K^\delta = \{z : d(z, K) < \delta\}, \quad d(z, w) = |z - w|(1 + |z|^2)^{-1/2}(1 + |w|^2)^{-1/2}. \quad (42)$$

The notation $K_n \rightarrow K$ stands for convergence of compacta K_n , $n = 1, 2, \dots$, to the compact K in the spherical Hausdorff metric. We remind that from each sequence of compacta $K_n \subset \overline{\mathbb{C}}$ one can extract a convergent subsequence.

We may suppose without loss of generality that the sequence $\{k_j\}_{j=1}^\infty$ of natural numbers is such that the sequences of continua $\{F_{k_j}^+\}_{j=1}^\infty$ and $\{F_{k_j}^-\}_{j=1}^\infty$ converge as $j \rightarrow \infty$ to some continua F^+ and F^- , respectively (if needed, we pass to subsequences). It can be easily seen that $F := (F^+ \cup F^-) \in \mathfrak{K}_{a_3, a_4}^{a_1, a_2}$, where $\mathfrak{K}_{a_3, a_4}^{a_1, a_2}$ denotes that set of compacta K , consisting of the union of continua $K^+ \cup K^-$, where $K^+ \subset D \cap \{\operatorname{Im} z \geq 0\}$ and connects the point a_1 with the point a_2 , whereas $K^- \subset D \cap \{\operatorname{Im} z \leq 0\}$ and connects a_3 with a_4 . We show that

$$\operatorname{cap} F = \inf_{K \in \mathfrak{K}_{a_3, a_4}^{a_1, a_2}} \operatorname{cap} K. \quad (43)$$

In fact, let

$$S_j^+ : \{\operatorname{Im} z \geq 0\} \mapsto \{\operatorname{Im} z \geq j^{-1}\} \quad \text{and} \quad S_j^- : \{\operatorname{Im} z \leq 0\} \mapsto \{\operatorname{Im} z \leq -j^{-1}\}$$

denote the transformation, determined in the following way:

$$S_j^+ z = \operatorname{Re} z + i \begin{cases} \operatorname{Im} z, & \text{if } \operatorname{Im} z > j^{-1}, \\ j^{-1}, & \text{if } 0 \leq \operatorname{Im} z \leq j^{-1}, \end{cases} \quad S_j^- z = \overline{S_j^+(\bar{z})}.$$

If $K = (K^+ \cup K^-) \in \mathfrak{K}_{a_3, a_4}^{a_1, a_2}$, then we let $S_j K = (S_j^+ K^+) \cup (S_j^- K^-)$. We easily see that if $K \in \mathfrak{K}_{a_3, a_4}^{a_1, a_2}$, then $S_j K \in \mathfrak{K}_{E_{k_j}, f^*}$ and $S_j K \rightarrow K$ as $j \rightarrow \infty$. Hence, from (38), in view of (3) and that $\lim_{j \rightarrow \infty} \rho_{k_j} = 1$, we obtain that for each $K \in \mathfrak{K}_{a_3, a_4}^{a_1, a_2}$

$$\operatorname{cap} K = \lim_{j \rightarrow \infty} \operatorname{cap} S_j K = \lim_{j \rightarrow \infty} \operatorname{cap}_{\psi_{k_j}} (S_j K) \geq \lim_{j \rightarrow \infty} \operatorname{cap}_{\psi_{k_j}} (F_{k_j}) = \lim_{j \rightarrow \infty} \operatorname{cap} F_{k_j} = \operatorname{cap} F,$$

From here, we derive the equality (43).

For $p = 4$ and $p = 5$ denote by L_p the intersection of L with the halfplane $\{\operatorname{Im} z \geq 2^{-p}\}$. Recall that $a_0 = \sum_{l=1}^4 a_l/4 = 2^{-4}i$. Therefore, L_4 is a union of the upper halfdiagonals $[a_1, a_0]$ and $[a_0, a_2]$ of the square with vertices at the points a_1, a_2, a_3, a_4 . We show that $L_5 \setminus F^+ \neq \emptyset$. In fact, otherwise $F^+ \supseteq L_5 \supsetneq L_4$, whence,

$$\operatorname{cap}(L_4 \cup F^-) < \operatorname{cap}(L_5 \cup F^-) \leq \operatorname{cap}(F^+ \cup F^-) = \operatorname{cap} F.$$

This inequality contradicts (43), thanks to the fact that $(L_4 \cup F^-) \in \mathfrak{R}_{a_3, a_4}^{a_1, a_2}$.

Thus, there is a point $z^* \in L_5 \setminus F^+$. Denote $\tilde{\mu}_{F_{k_j}}$ the balayage of the measure μ_{k_j} onto F_{k_j} . Without loss of generality, we may suppose that the sequence of measures $\{\tilde{\mu}_{F_{k_j}}\}_{j=1}^\infty$ (passing to subsequences, if needed) converges in the weak sense

$$\tilde{\mu}_{F_{k_j}} \xrightarrow{*} \mu_F^*, \quad j \rightarrow \infty. \quad (44)$$

Since $F_{k_j} \rightarrow F$, we have $\text{supp } \mu_F^* \subset F$. From $z^* \in L_5 \setminus F^+ \subset L \setminus F$, it follows that $\mu_F^* \neq \lambda_L^{\text{eq}}$, where λ_L^{eq} is the equilibrium measure of the compactum L (in the absence of an external field).

Fix some (E, μ) -table of points of interpolation $\tilde{E}_n = \{e_{2n+1, l}\}_{l=1}^{2n+1}$, $n = 1, 2, \dots$. Then $k_j \tilde{E}_n = \{k_j e_{2n+1, l}\}_{l=1}^{2n+1}$, $n = 1, 2, \dots$, is an (E_{k_j}, μ_{k_j}) -table.

We get from our assumption (39), after applying Theorem 1 and the Corollary of Gonchar–Rakhmanov's theorem, that for all $j = 1, 2, \dots$

$$\frac{1}{n} \delta_{Q_n^{k_j}} \xrightarrow{*} \tilde{\mu}_{F_{k_j}}, \quad n \rightarrow \infty, \quad (45)$$

where $Q_n^{k_j}$ is the denominator of the Padé approximants of the function $f^* \in \mathcal{A}(E_{k_j})$, constructed with respect to the (E_{k_j}, μ_{k_j}) -table of the interpolation nodes $k_j \tilde{E}_n$.

Let $C(D)$ be the space of all continuous functions on D . Select some countable and everywhere dense set of functions g_1, g_2, \dots in $C(D)$; write, for $\nu_1, \nu_2 \in M(D)$

$$\rho(\nu_1, \nu_2) := \sum_{p=1}^{\infty} \frac{1}{2^p \max_{z \in D} |g_p(z)|} \left| \int g_p d\nu_1 - \int g_p d\nu_2 \right|.$$

We note that

$$\rho(\nu_1, \nu_3) \leq \rho(\nu_1, \nu_2) + \rho(\nu_2, \nu_3). \quad (46)$$

Moreover, in view of (5) (the definition of weak convergence), for $\nu_j, \nu \in M(D)$

$$\nu_j \xrightarrow{*} \nu, \quad j \rightarrow \infty \iff \lim_{j \rightarrow \infty} \rho(\nu_j, \nu) = 0. \quad (47)$$

In particular, relations (44) and (45) can be rewritten as equalities; namely,

$$\lim_{j \rightarrow \infty} \rho(\tilde{\mu}_{F_{k_j}}, \mu_F^*) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \rho\left(\frac{1}{n} \delta_{Q_n^{k_j}}, \tilde{\mu}_{F_{k_j}}\right) = 0. \quad (48)$$

Denote $J_{n,j}$ the set all indices $l \in \{1, \dots, 2n+1\}$ such that $|e_{2n+1,l}| > k_j^{-1/2}$, and let $|J_{n,j}|$ be it's cardinality. We get from the definition of (E, μ) -table that for all $n \geq n(j)$

$$|J_{n,j}|/(2n) \geq \mu(E \setminus [-k_j^{-1/2}, k_j^{-1/2}]) - k_j^{-1}. \quad (49)$$

Along with the sequence $\{k_j\}_{j=1}^\infty$ that we already constructed, we introduce a sequence $\{n_j\}_{j=1}^\infty$ of natural numbers according to the following scheme. We fix arbitrarily a starting number $n_1 \in \mathbb{N}$. Suppose that the numbers n_1, \dots, n_{j-1} have already been found, and select the number n_j in such a way that $n_j > \max\{n_{j-1}, n(j)\}$ and

$$\rho\left(\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}}, \tilde{\mu}_{F_{k_j}}\right) \leq k_j^{-1}. \quad (50)$$

Such a choice of the number n_j is possible thanks to the second equality in (48).

Hence, the numbers k_j and n_j are defined for all $j = 1, 2, \dots$. Set $n_j^* = |J_{n_j,j}|$; taking into account the obvious inequality $|J_{n_j,j}| \leq 2n_j + 1$, we obtain that inequality (49) and condition (18) lead to

$$\lim_{j \rightarrow \infty} n_j^*/n_j = 2. \quad (51)$$

Let

$$T_{n_j}(z) = \prod_{l=1}^{2n_j+1} \left(1 - \frac{z}{k_j e_{2n_j+1,l}}\right), \quad \omega_{n_j}(z) = \prod_{l \in J_{n_j,j}} \left(1 - \frac{z}{k_j e_{2n_j+1,l}}\right).$$

We note that $\deg \omega_{n_j} = n_j^*$, the polynomial ω_{n_j} divides T_{n_j} , is zero-free for $|z| \leq k_j^{1/2}$ and for each $z \in D$, the inequalities

$$\left(1 - k_j^{-1/2}\right)^{n_j^*} \leq |\omega_{n_j}(z)| \leq \left(1 + k_j^{-1/2}\right)^{n_j^*} \quad (52)$$

hold. Thus, the functions $\Psi_{n_j}(z) = 1/\omega_{n_j}(z)$, $j = 1, 2, \dots$, are holomorphic in the disk D and

$$\frac{1}{n_j} \log |\Psi_{n_j}(z)| \rightrightarrows 0 = \mathcal{V}^{-\delta_\infty}(z), \quad z \in D. \quad (53)$$

We remark that the compactum L is symmetric in the external field $\mathcal{V}^{-\delta_\infty} \equiv 0$ (in other words, in the absence of an external field), the complement $\overline{\mathbb{C}} \setminus L$ is connected and $f_L \in \mathcal{H}_0(\overline{\mathbb{C}} \setminus L)$. From the above consideration it follows that the compact set L , the neighborhood $\Omega = \{|z| < 1\}$ of L , the measure $\mu = \delta_\infty$, the sequence of functions Ψ_{n_j} , and the function f_L satisfy conditions 1°–4° in Gonchar–Rakhmanov’s theorem (after replacing the entire set of natural numbers by the sequence $\{n_j\}_{j=1}^\infty$). We later return to this version of Gonchar–Rakhmanov’s theorem, a bit stronger than the original. Before applying this statement we check whether the polynomials $Q_{n_j}^{k_j}$ of degree not greater than n_j satisfy the following orthogonality conditions:

$$\oint_L Q_{n_j}^{k_j}(t) \Psi_{n_j}(t) f_L(t) t^\nu dt = 0, \quad \nu = 0, 1, \dots, n_j^* - n_j - 2 \quad (j = 1, 2, \dots); \quad (54)$$

where we integrate along a contour encircling L and close enough to it.

In fact, $f_L(z) = f^*(z) = f_{F_{k_j}}(z)$ for all $z \in \overline{\mathbb{C}} \setminus D$, $j \geq \max\{j_0, j_1\}$, where $f_{F_{k_j}}$ is the branch of f holomorphic in the domain $\overline{\mathbb{C}} \setminus F_{k_j}$ and determined by the condition $f_{F_{k_j}}(\infty) = 1$. By definition of Padé approximants of the function $f_{F_{k_j}} \in \mathcal{A}(E_{k_j})$, the difference $(Q_{n_j}^{k_j} f_{F_{k_j}} - P_{n_j}^{k_j})(z)$ vanishes at all zeros of the polynomial $T_{n_j}(z)$ and, in particular, at all zeros of the polynomial $\omega_{n_j}(z)$. Because of the fact that all zeros of $\omega_{n_j}(z)$ belong to the set $\{|z| > k_j^{1/2}\} \subset \overline{\mathbb{C}} \setminus D$, on which $f_{F_{k_j}} = f_L$, we come to the conclusion that

$$\frac{Q_{n_j}^{k_j} f_L - P_{n_j}^{k_j}}{\omega_{n_j}} \in \mathcal{H}(\overline{\mathbb{C}} \setminus L). \quad (55)$$

The left hand side of the inclusion (55) is of order $O(1/z^{n_j^* - n_j})$ as $z \rightarrow \infty$. Therefore, for all $\nu = 0, 1, \dots, n_j^* - n_j - 2$

$$\oint_L \frac{Q_{n_j}^{k_j} f_L - P_{n_j}^{k_j}}{\omega_{n_j}}(t) t^\nu dt = 0. \quad (56)$$

Since all zeros of the polynomial $\omega_{n_j}(z)$ are lying outside of $D \supset L$, we get

$$\oint_L \frac{P_{n_j}^{k_j}}{\omega_{n_j}}(t) t^\nu dt = 0, \quad \nu = 0, 1, \dots \quad (57)$$

Hence, equalities (56) can be rewritten as follows:

$$\oint_L \frac{Q_{n_j}^{k_j} f_L}{\omega_{n_j}}(t) t^\nu dt = 0, \quad \nu = 0, 1, \dots, n_j^* - n_j - 2,$$

which coincide with (54).

Now, let us address to the relations (54) and (15). Notice that in (54) the indices $\nu = n_j^* - n_j - 1, \dots, n_j - 1$ are absent. It follows from (51) that the number $2n_j - n_j^* + 1$ of absent relations does not exceed $o(n_j)$. Taking this into account and repeating verbatim the proof elaborated by Gonchar and Rakhmanov, we see that statement (i) of Gonchar–Rakhmanov’s theorem follows from relation (54) just as from relation (15). Consequently, by the stronger version of statement (i) in Gonchar–Rakhmanov’s theorem,

$$\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}} \xrightarrow{*} \lambda_L^{\text{eq}}, \quad j \rightarrow \infty, \quad (58)$$

where λ_L^{eq} stands for the equilibrium measure of the compactum L (in the absence of an external field).

On the other hand, inequality (46) yields that

$$\rho\left(\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}}, \mu_F^*\right) \leq \rho\left(\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}}, \tilde{\mu}_{F_{k_j}}\right) + \rho(\tilde{\mu}_{F_{k_j}}, \mu_F^*).$$

Now, considering inequality (50) and the first equality in (48), we arrive at the equality

$$\lim_{j \rightarrow \infty} \rho\left(\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}}, \mu_F^*\right) = 0,$$

which is equivalent to the relation

$$\frac{1}{n_j} \delta_{Q_{n_j}^{k_j}} \xrightarrow{*} \mu_F^*, \quad j \rightarrow \infty. \quad (59)$$

As noticed before, $\lambda_L^{\text{eq}} \neq \mu_F^*$. Thus, (59) contradicts (58). Therefore, the proof of Theorem 2 is complete.

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